



Partial Derivatives: Extreme Values and Saddle Points (Modified)

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Article information

Article history:

Received: May, 02, 2026

Accepted: June, 04, 2026

Available online: June, 06, 2026

Keywords:

Partial derivative,
Point of inflection,
Saddle point

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Citation : Abdul Kareem, S. (2026). Partial Derivatives : Extreme Values and Saddle Points (Modified): Partial Derivatives : Extreme Values and Saddle Points (Modified). Journal of Advanced Sciences and Nanotechnology, 1(1), 1–6.

Abstract

One of the most useful applications for derivatives of a function of maximum and/or minimum values. This application is also important for functions of two or more variables". The main ideas of finding critical points and Using derivatives tests are still valid.^[1] The tests of second derivatives for one variable give us the results for (a) Maximum and/or minimum points, (b) Points of inflection and (c) Test is inconclusive. while the second derivative for two variables give us the result for^[2] : (a) Maximum and/or minimum points, (b) Saddle point and (c) Test is inconclusive. this paper had concentrated about the test of second derivative for the functions have two variables which depend on : f_{xx} f_{yy} f_{xy} in such a way that there is a relationship between the discriminant (D) and f_{xx} . modification will give relationship between D and f_{yy} instead of f_{xx} and the similarity between the point of infection in normal derivative and the saddle point in partial derivatives.

DOI: <https://doi.org/10.55945/joasnt.2026.1.1.1-6>,

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1. Introduction

A partial derivative measure the rate of change of multivariable function with respect to one variable while holding all others constant representing the slope in a specific direction. it allows for isolating the effect of a single input on the output, often used to optimize systems. there are first and second derivatives such as f_x , f_y , f_{xx} , f_{yy} . The extrem values and saddle points are related with both first and second partial derivatives. To the test of the second derivative for the functions of two variables $f(x, y)$, there is a relationship between the determinant:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

2. Theoretical Part

2.1 Normal Derivative

The Derivative of a function :The derivative of the function f with respect to the variable x is the function f' whose value at x is :

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exist.^[3] Notation : There are many ways to denote the derivative of a function:

$$[Y', \frac{dy}{dx}, \frac{df}{dx}, f'(x)]$$

These are called the (First derivative of the function). Also we have the (second derivative of the function) denoted as:

$$[y'', \frac{d^2y}{dx^2}, \frac{d^2f}{dx^2}, f''(x)]$$

There are many higher derivatives, but now we related with first and second derivatives only for the purpose of this paper and the relation between them.

2.2 Extreme values of functions :

A-1 : The first derivative test for local extreme values^[4] "A function (f) may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of (f') in the point immediate vicinity. As x moves from left to right, the values of (f) increase where $f' > 0$ and decrease where $f' < 0$.

At a point where,

$$\frac{dy}{dx} > 0,$$

the tangent to the graph of $y = f(x)$ has positive slope, So the Linearization of (f) at such point is an (increasing function) and

$$\text{if } \frac{dy}{dx} < 0,$$

the tangent line slope downward and the curve falls, so the linearization of (f) at such point is (decreasing function). Also, it is evident that regions of rise and fall are usually separated by high or low points.

where $\frac{dy}{dx}$ is zero and the tangent is horizontal.

"If $f(x_1) > f(x_2)$ when $x_1 > x_2$ for all x_1 and x_2 in $[a, b]$ we say that (f) is an (increasing) function on $[a, b]$. If $(x_1) < f(x_2)$ when $x_1 < x_2$ for all x_1 and x_2 in $[a, b]$ we say that (f) is a (decreasing function) on $[a, b]$.

When $\frac{dy}{dx}$ is positive, the curve is (rising) to the right.

When $\frac{dy}{dx}$ is negative, the curve is (falling) to the right

When $\frac{dy}{dx} = 0$, we have high or low points".^[2,3]

A-2: Maxima and Minima Theory^[5]

a- If $f(x)$ is a continuous function in its domain, then at least one maximum or one minimum should lie between equal values of $f(x)$.

b- Maxima and minima occur alternately i.e. between two minima, there is one maxima and vice versa.

c- If $f(x)$ tends to infinity as x tends to a or b and $f'(x) = 0$ only for one value of x , i.e., (c) between (a) and (b) then $f(c)$ is the minimum and the least value.

If $f(x)$ tends to $(-\infty)$ as x tends to a or b , then $f(c)$ is the maximum and the highest value.

A-3: End Point Extrema^[6]:

If the domain of (f) is the bounded, closed interval $[a, b]$ and $f'(a+)$ and $f'(b-)$ exist, then it is easy to verify that:

$$\begin{aligned} & \text{(f) has a local } \begin{cases} \text{maximum} \\ \text{minimum} \end{cases} \text{ , at (a) if } \begin{cases} f'(a) < 0 \\ f'(a) > 0 \end{cases} \\ & \text{and (f) has a local } \begin{cases} \text{minimum} \\ \text{maximum} \end{cases} \text{ , at (b) if : } \begin{cases} f'(b-) < 0 \\ f'(b-) > 0 \end{cases} \end{aligned}$$

A-4 : Significance of the sign of the second derivative^[7]:

The sign of the second derivative tells whether the graph $y = f(x)$ is concave upward if (y'' positive) or downward if (y'' negative), Let (f) be differentiable on an interval (I).The graph of (f) is concave up on (I) if (f') is increasing. The graph of (f) is concave down on (I) if (f') decreasing. If (f') is constant then the graph is said to have no concavity.

A-5 : Points of Inflection :^[8]

A point where the curve changes the direction of its concavity from downward to upward or vice versa, is called a (point of inflection).

Inflection points characterized by a change in the sign of $\frac{d^2y}{dx^2}$, such a change of sign may occur where (a) $\frac{d^2y}{dx^2} = 0$, (b) fails to exist

B- : Partial Derivatives :^[9]

The calculus of several variables is similar to single variable calculus applied to several variables one at time. When we hold all but one of the independent variables of function constant and differentiate with respect to that one variable, we get a "partial" derivatives of (f) with respect to x at the point (x₀, y₀) as the ordinary derivatives of f(x, y) with respect to x at the point x = x₀. To distinguish partial derivatives from ordinary derivatives we use the symbol (∂) rather than (d). In general: to find the local extreme values of a function of a single variables f(x) we look for points where the graph has a horizontal tangent line at such points we then look for: Local maxima, Local minima, Points of inflection. For function f(x, y) of two variables, we look for points where surface z = f(x, y) has a horizontal tangent plane, at such points we look for : Local maxima, Local minima, Saddle points.

B-1 : Definition^{[10],[11]} : The partial derivative of f(x, y) with respect to x at the point

$$(x_0, y_0) \text{ is : } \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \text{ Provided The limit exist.}$$

Also : The partial derivative of f(x, y) with respect to y at the point

$$(x_0, y_0) \text{ is } \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \text{ Provided The limit exist.}$$

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x : $\frac{\partial f}{\partial x}(x_0, y_0)$, $f_x(x_0, y_0)$ also $\frac{\partial f}{\partial y}(x_0, y_0)$, f_y

B-2 : Second - order partial derivatives :when we differentiate a function f(x, y) twice, we produce its second – order derivatives, these are denoted by :

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

Although there are higher order partial derivatives.

B-3 : The mixed derivative Theorem (Clairaut's Theorem)^[3]:

$$f_{xy} = f_{yx}$$

B-4 : Extreme Values and Saddle Points^[12] :

Let f(x, y) be defined on a region (R) containing the point (a, b), Then : a- f(a, b) is a (local maximum) value of (f) if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

b- f(a, b) is a (Local minimum) value of (f) if $f(a, b) \leq f(x, y)$ in an open disk centered at (a, b). The extreme values of f(x, y) can occur only at : i) boundary points of domain of (f) and end points. ii) Critical points (interior points) when $f_x = f_y = 0$ or points where f_x or f_y fail to exist. Local extrema are also called (Relative extrema).

B-5 : First derivative test for local extreme values^[13]: If f(x, y) has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivative exist there then $f_x(a, b) = 0$ and $f_y(a, b) = 0$ thus a first necessary condition for a maximum or a minimum is :

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ at } (a, b)$$

In terms of differentials, this means that $df = 0$ at (a, b). Points at which this occur are called (stationary points of the function) and the values of the function at those points are called (stationary values).

B-6 : Saddle Points :^{[14],[15]} These are stable points where the function has a local maximum in one direction, but a local minimum in another direction so, a saddle point is neither a local maximum nor a maximum nor a

B-7 : Second derivative test for extreme values:^{[15],[16]}

Suppose $z = f(x, y)$ has a critical point at (x₀, y₀) and has continuous second partial D (discriminant) as :

$$D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Then :

- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ (concave down) then f(x₀, y₀) is a relative maximum
- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ (concave up) then f(x₀, y₀) is a relative minimum
- If $D < 0$ then f(x₀, y₀) is a saddle point
- If $D = 0$, the test is inconclusive, and we must examine the critical point using other means

4- Modification to the theory of second derivative test :

i- derivative test for extreme values : as we see the test for local maximum and local minimum is between (D) and f_{xx} as below :

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

- If $D > 0$ and $f_{xx} < 0 \therefore$ local maximum.
- If $D > 0$ and $f_{xx} > 0 \therefore$ local minimum.

I suggest the test is between D and f_{yy} and the equations become :

- If $D > 0$ and $f_{yy} < 0 \therefore$ local maximum.
- If $D > 0$ and $f_{yy} > 0 \therefore$ local minimum.

ii- Saddle Point and Points of Inflection: Saddle Points and Points of Inflection have the same criteria, where these points between maximum points and minimum points, so they are not local maximum not. Local minimum. Both saddle points and points of inflection depend on the second derivative test.

In points of inflection we put $\frac{d^2y}{dx^2} = 0$ While in saddle point $f_{xx} f_{yy} - (f_{xy})^2 < 0$

So :Both saddle points and the points of inflection should have the same result of the second Derivative, i.e : points of inflection,

$$\frac{d^2y}{dx^2} = 0$$

saddle point $f_{xx} f_{yy} - (f_{xy})^2 = 0$ Or, point of inflection, $\frac{d^2y}{dx^2} < 0$, saddle point $f_{xx} f_{yy} - (f_{xy})^2 < 0$

This optional section should be used only if more extensive theoretical derivations are needed. Simpler theories and methods should be a part of either Introduction or Experiment, respectively. All equations, including those describing chemical reactions, must be written in separate lines and numbered. The symbols of quantities should be explained immediately below the equation if they were used for the first time. Theoretical part should be Times New Roman, justified, regular; font size: 11 single. If you have any figures or tables in this section, please use the same format that will be mentioned in the "Results and discussion" part.

3. Experimental Procedure

ex 1 : $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2 = 0 \rightarrow y = 2x + 2$ (1)

$f_y = x - 2y - 2 = 0$ (2) Sub (1) in (2) :

$x - 2(2x + 2) - 2 = 0$

$x - 4x - 4 - 2 = 0 \rightarrow -3x = 6 \rightarrow x = -2$

$y = 2(-2) + 2 = -2 \therefore$ The point $(-2, -2)$

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-2) - (1)^2 = 3$

$D > 0$, $f_{yy} < 0 \rightarrow$ Local maximum

$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f(-2, -2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$

$f(-2, -2) = 8$ Local maximum

ex 2: $f(x, y) = 3y^2 - 2y^2 - 3x^2 + 6xy$

$f_x = -6x + 6y = 0 \rightarrow x = y$ (1)

$f_y = 6y - 6y^2 + 6x = 0 \div 6$

$y - y^2 + x = 0$ (2) sub (1) in (2)

$x - x^2 + x = 0 \rightarrow 2x - x^2 = 0$

$x(2 - x) = 0 \rightarrow x = 0$, $y = 0$ (0, 0), $x = 2$, $y = 2$ (2, 2)

$f_{xx} = -6$, $f_{yy} = 6 - 12y$

when $y = 0$, $f_{yy} = 6$

when $y = 2$ $f_{yy} = 6 - 24 = -18$

$f_{xy} = 6$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (-6)(6) - (6)^2 = -36 - 36 = -72 < 0$, (0, 0) Saddle point

$D = (-6)(-18) - (6)^2 = 72 > 0$, $f_{yy} = -18 < 0$ (2, 2) maximum point

$f(x, y) = 3y^2 - 2y^2 - 3x^2 + 6xy$

$f(0, 0) = 0$ Saddle point

$f(2, 2) = 3(4) - 2(8) - 3(4) + 6(2)(2)$

$$f(2)(2) = -16 + 24 = +6$$

(2, 2, 6) maximum point

Note : from this example we had seen that f_{yy} has two different values while f_{xx} has one value only.

$$\text{ex 3 : } f(x, y) = y^2 + xy - 2x - 2y + 2$$

$$f_x = y - 2 = 0 \rightarrow y = 2$$

$$f_y = 2y + x - 2 = 0$$

$$2(2) + x - 2 = 0 \rightarrow x = -2$$

The point is (-2, 2), $f_{xx} = 0$, $f_{xy} = 1$, $f_{yy} = 2$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (0)(2) - (1)^2 = -1$$

$D < 0$ saddle point

$$f(x, y) = y^2 + xy - 2x - 2y + 2$$

$$f(-2, 2) = (2)^2 + (-2)(2) - 2(-2) - 2(2) + 2 \\ = 4 - 4 + 4 - 4 + 2 = 2$$

$\therefore f(-2, 2, 2)$ Saddle point

$$\text{ex 4 : } f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

$$f_x = 4x + 3y - 5 = 0 \quad \dots\dots\dots (1) \quad (3)$$

$$f_y = 3x + 8y + 2 = 0 \quad \dots\dots\dots (2) \quad (4)$$

$$-12x + 9y + 15 = 0$$

$$12Xx + 32y + 8 = 0$$

$$12Xx + 32y + 8 = 0 \rightarrow y = -1$$

Subin (1) : $4x + 3(-1) - 5 = 0 \rightarrow x = 2$. The point is (2, -1)

$$f_{xx} = 4, f_{xy} = 3, f_{yy} = 8$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (4)(8) - (3)^2 \\ = 23 > 0 \quad f_{yy} > 0 \text{ Local minimum}$$

$$f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

$$f(2, -1) = 2(2)^2 + 3(2)(-1) + 4(-1)^2 - 5(2) + 2(-1) \\ = 8 - 6 + 4 - 10 - 2 = -6$$

$f(2, -1, -6)$ Local minimum

4. Results and Discussion

For the second partial derivative test for local extreme and saddle point, we can use f_{yy} with the discriminant (D) in the same manar as use f_{xx} with (D). for maximum and minimum test for $f(x, y)$ which has a critical point at (x_0, y_0) and has a discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$ we can said :

- If $D > 0$ and $f_{yy} < 0$ then the point is local maximum
- If $D > 0$ and $f_{yy} > 0$ then the point is local minimum

And as seen in example sometimes the use of (f_{yy}) is more useful than f_{xx} in the test for extreme values (Local maximum, Local minimum).ii = for the saddle point : Saddle point in $f(x, y)$ and points of inflection in $f(x)$ have the same attribute,so I suggest that should have the same mathematical meaning in the form :

- Either both should be less than zero $\frac{d^2y}{dx^2} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 < 0$
- or both should be equal to zero $\frac{d^2y}{dx^2} = 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 0$

5. Conclusions

The Conclusions section contains a brief analysis of principal findings and significance of the work. The Conclusions must not be a re-worded abstract. Unlike an abstract, the Conclusions are primarily determined to those who read the whole paper. Conclusions should be Times New Roman, justified, regular; font size: 11 single.

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